


# A New Formulation of Maxwell's Equations

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**Abstract:** In this paper, new forms of Maxwell's equations in vector and scalar variants are presented. The new forms are based on the use of Gauss's theorem for magnetic induction and electrical induction. The equations are formulated in both differential and integral forms. In particular, the new forms of the equations relate to the non-stationary expressions and their integral identities. The indicated methodology enables a thorough analysis of non-stationary boundary conditions on the behavior of electromagnetic fields in multiple continuous regions. It can be used both for qualitative analysis and in numerical methods (control volume method) and optimization. The last Section introduces an application to equations of magnetic fluid in both differential and integral forms.

**Keywords:** Maxwell's equations; divergence theorem; integral form; magnetism; optimization; analysis

## 1. Introduction

Magnetic and magnetorheological fluids have been increasingly used in recent years [1–3]. Their properties, especially viscosity, can be significantly changed by the effects of the magnetic field [4–10]. The mathematical model is, in such cases, composed of Maxwell's and Navier–Stokes equations [11], which are mostly solved by numerical methods [12–15]. One of these models is presented at the end of this paper. The solution behavior of the mentioned models can be assessed on the basis of the symmetry or asymmetry of the operators forming the mathematical model [16,17]. Let us write, for example, Maxwell's equations in the following operator form:

$$\left\| \begin{array}{cc} -\frac{\partial}{\partial t} & \text{curl} \\ \text{div} & 0 \end{array} \right\| \cdot \left\| \begin{array}{c} D \\ H \end{array} \right\| = \left\| \begin{array}{c} j \\ \rho \end{array} \right\| \quad (1)$$

$$\left\| \begin{array}{cc} \frac{\partial}{\partial t} & \text{curl} \\ \text{div} & 0 \end{array} \right\| \cdot \left\| \begin{array}{c} B \\ E \end{array} \right\| = 0 \quad (2)$$

From here, the symmetry of the problem for a nonconductive environment, where  $j = 0$ ,  $\rho = 0$ , is quite obvious. Furthermore, it follows that the non-stationary field is rotational, because it is not possible to set  $E = \text{grad}\phi$ . It is therefore necessary to use new methods for non-stationary problems solution. The operator equations, written in the summation symbolics, also show that

$$(\text{curl}E)_i = \varepsilon_{ijk} \frac{\partial E_k}{\partial x_j} \quad (3)$$

The expression  $\frac{\partial E_k}{\partial x_j}$  can be decomposed into a symmetric  $E_{kj} = \frac{1}{2} \left( \frac{\partial E_k}{\partial x_j} + \frac{\partial E_j}{\partial x_k} \right)$  and



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an antisymmetric part  $\widehat{E}_{kj} = \frac{1}{2} \left( \frac{\partial E_k}{\partial x_j} - \frac{\partial E_j}{\partial x_k} \right)$ .

While the Levi–Civita tensor  $\varepsilon_{ijk}$  is antisymmetric, only the antisymmetric part  $\widehat{E}_{kj}$  is used in Expression (3). Thus:

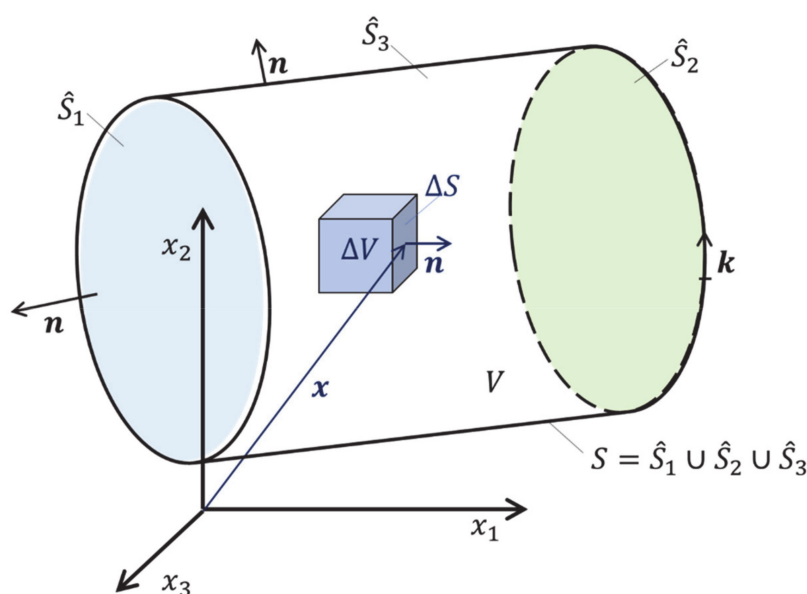
$$(\text{curl} E)_i = \varepsilon_{ijk} \widehat{E}_{kj} \quad (4)$$

These modifications could be deepened by a more detailed study of the symmetry of the mathematical model operators depending on the boundary conditions. A new formulation of Maxwell's equations, which is essentially based on Gauss's divergence theorem [17], can also contribute to this analysis.

According to the authors, the mathematical formulation of Gauss's divergence theorem is underestimated in terms of the physical substantiality of the problem. Gauss's divergence theorem explains the following important finding: every spatial change in the physical variable  $f(x, t)$ , regardless of its tensor character, has its response at the boundary of a closed region:

$$\int_V \frac{\partial f(x, t)}{\partial x_i} dV = \int_S f(x, t) n_i dS \quad (5)$$

where  $f$  can be understood as a symbolic variable. Of course, the term (5) has the opposite meaning. If we act on the boundary of the region  $S$  with the effect of the variable  $f$ , this variable changes within the volume  $V$ . In brief, every change within the volume  $V$  can be measured at the boundary  $S$ , where it is reflected in its functional value; see Figure 1.



**Figure 1.** General volume of the liquid  $V$  surrounded by continuous closed area  $S$  that consists of open areas  $\hat{S}$  (outlined by closed curve  $k$ ). Volume  $V$  can be split into control volumes  $\Delta V$ , closed by control areas  $\Delta S$ .

This principle can be used in qualitative analysis, numerical methods, and optimization of the electromagnetic field. The main reason is that this principle gives a very good idea, both qualitative and quantitative, of the influence of boundary conditions on the problem, which assignment is often in hands of the researcher.

After all, in electromagnetic fields solution, the engineering practice successfully uses integral identities both on the principle of Gaussian divergence theorem for the closed area  $S$  and Stokes theorem for the open area  $\hat{S}$  bounded by a curve  $k$  [15–19].

For example:

$$\text{curl} E + \frac{\partial B}{\partial t} = 0; \quad \int_k E dk + \int_{\hat{S}} \frac{\partial B}{\partial t} \cdot n d\hat{S} = 0 \quad (6)$$

and others [20]. Since this paper is focused on solving non-stationary problems, it is appropriate to modify Maxwell's equations into a more suitable form for optimization, qualitative analysis, and numerical methods. This can be achieved by using Gauss's divergence theorem and the symmetry of the Kronecker delta operator ( $\delta_{ij} = \delta_{ji}$ ). The adjustment applies to all non-stationary terms of Maxwell's equations for non-conductive media and  $\frac{\partial \mathbf{B}}{\partial t}$  for conductive media, where  $j \neq 0$ ,  $\rho \neq 0$ . The modification, the proof of which is given in Section 2, can be expressed by a non-stationary term in a more suitable form for the application of Gauss's divergence theorem:

$$\frac{\partial B_i}{\partial t} = \frac{\partial}{\partial x_j} \left( \frac{\partial B_i}{\partial t} x_j \right) \quad (7)$$

The scalar variant of Maxwell's equations (Section 4) is also presented in the work, where new functions are introduced:

Modified intensity of imprinted forces:

$$\hat{E}[V \cdot m^{-1}] = \frac{\partial \mathbf{B}}{\partial t} \mathbf{x} \quad (8)$$

Modified stress of printed forces:

$$\hat{U}[V] = \frac{\partial \mathbf{B}}{\partial t} |\mathbf{x}|^2 \quad (9)$$

Based on Gauss's divergence theorem, they are again modified to a form suitable for analysis:

$$\frac{\partial B_i}{\partial t} x_i = \frac{\partial}{\partial x_j} \left( \frac{\partial B_i}{\partial t} x_i x_j \right) \quad (10)$$

Based on (7), by integrating (6) over the domain  $V$  surrounded by the surface  $S$ , new integral identities can be found, this time over the closed surface, where all boundary conditions appear.

$$\int_S (\mathbf{n} \times \mathbf{E}) dS + \int_S \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) \mathbf{x} dS = \mathbf{0} \quad (11)$$

The proof is given in Section 2. It follows from the above-mentioned equations that the article focuses on the appropriate modification of Maxwell's equations so that the influence of non-stationary terms in the field  $V$  is expressed by their values on the boundary  $S$  of the closed region. The solution is based on the use of symmetry conditions and Gauss's divergence theorem. The use of this procedure for qualitative analysis, numerical methods, and optimization are presented in the individual Sections for both the vector variant and the scalar variant of Maxwell's equations. Both conductive and non-conductive environments are considered in the solution. The last Section presents a mathematical model of the interaction of a magnetorheological fluid with a magnetic field. Even for this interdisciplinary problem, Gauss's divergence theorem can be used to redefine the mathematical model of Navier–Stokes equations. An example is given in Section 6.

A special part is devoted to the finite volume method for non-stationary problems [13–22]. In the classical method, a non-stationary term is identified through the control volume in terms of the mean values of the integral calculus. This method does not allow the use of the finite volume method, while the new variant will allow it; see Section 2.

## 2. Symmetry in Principles of the Solution

In the technical sciences, symmetry conditions play a special role, especially in the stability conditions of the system [13,14,23]. They have the same importance in the solution

of electromagnetism tasks, where we can find in the term  $\frac{\partial B_i}{\partial x_j}$  and  $\frac{\partial D_i}{\partial x_j}$ . These terms conclude the symmetric  $S_{ij}$  and antisymmetric  $A_{ij}$  part. It can be written:

$$\frac{\partial B_i}{\partial x_j} = S_{ij} + A_{ij}, \text{ where} \quad (12)$$

$$S_{ij} = \frac{1}{2} \left( \frac{\partial B_i}{\partial x_j} + \frac{\partial B_j}{\partial x_i} \right) \text{ and } A_{ij} = \frac{1}{2} \left( \frac{\partial B_i}{\partial x_j} - \frac{\partial B_j}{\partial x_i} \right)$$

Based on the symmetry principles, it is possible to find new shapes of Maxwell's equations using the Gauss divergence theorem. The principle can be explained, for example, on magnetic induction. Let us consider the following equations:

$$\text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \quad \text{div} \mathbf{B} = 0. \quad (13)$$

The same formulated in the index symbolic (Einstein summation symbolics):

$$\varepsilon_{ijk} \frac{\partial E_k}{\partial x_j} = -\frac{\partial B_i}{\partial t}; \quad \frac{\partial B_i}{\partial x_i} = 0. \quad (14)$$

Einstein's summation symbolic is used in the mentioned relation and the following text. We note that these relations depend on the antisymmetric operator  $\varepsilon_{ijk}$  and the expression  $\frac{\partial E_k}{\partial x_j}$ , which can be decomposed into symmetric and antisymmetric parts (see Section 1). Therefore, in the left part of Equation (14), only the antisymmetric part of the expression  $\frac{\partial E_k}{\partial x_j}$  manifests.

After these remarks, let us proceed to a derivation of a new variant of the equation  $\varepsilon_{ijk} \frac{\partial E_k}{\partial x_j} = -\frac{\partial B_i}{\partial t}$  by modifying its right part containing a non-stationary term. We start from the validity of the equation  $\frac{\partial B_i}{\partial x_i} = 0$ . For this purpose let us put:

$$\begin{aligned} \frac{\partial B_j}{\partial x_j} x_i &= 0; \\ \int_V \frac{\partial B_j}{\partial x_j} x_i dV &= \int_S B_j x_i n_j dS - \int_V B_j \frac{\partial x_i}{\partial x_j} dV = 0 \end{aligned} \quad (15)$$

The relationship uses per partes integration in 3D space. After using a Kronecker delta symmetry adjustment:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \delta_{ji} \quad (16)$$

Thus,  $B_j \frac{\partial x_i}{\partial x_j} = B_i$ . It is possible to write Equation (15) in the shape:

$$\int_V B_i dV = \int_S B_j x_i n_j dS; \text{ or in the vector form} \quad (17)$$

$$\int_V \mathbf{B} dV = \int_S (\mathbf{B} \cdot \mathbf{n}) \mathbf{x} dS \quad (18)$$

Because it also holds that  $\frac{\partial}{\partial x_j} \left( \frac{\partial B_j}{\partial t} \right) = 0$ , it can be derived by analogy:

$$\int_V \frac{\partial B_i}{\partial t} dV = \int_S \frac{\partial B_j}{\partial t} x_i n_j dS; \text{ or in the vector form} \quad (19)$$

$$\int_V \frac{\partial \mathbf{B}}{\partial t} dV = \int_S \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) \mathbf{x} dS \quad (20)$$

Expressions (19) and (20) are very important, because they point out the fact that unsteady states of the magnetic flux density are generated on the borders of the area and vice versa. From the Expression (17) follows the next important result based on the divergence theorem:

$$\frac{\partial B_i}{\partial t} = \frac{\partial}{\partial x_j} \left( \frac{\partial B_j}{\partial t} x_i \right) \quad (21)$$

Using Expression (21), it is possible to correct the form of Equation (14) on the principle of symmetry. After the substitution of the above-mentioned criteria, we obtain a new shape of Maxwell equations.

$$\frac{\partial}{\partial x_j} \left( \varepsilon_{ijk} E_k + \frac{\partial B_j}{\partial t} x_i \right) = 0 \quad (22)$$

In Equation (22), the first term in brackets represents the antisymmetric tensor of second grade, and in the second term, it is possible to decompose into the symmetric and antisymmetric parts.

Equation (19) can be used to modify the control volume method.

In the current control volumes method, the integration of the non-stationary term given in Maxwell's equations is expressed on the basis of the mean value of the integral calculus [5,11–14,20,22,24]. Thus, in the form (see Figure 1):

$$\int_V \frac{\partial \mathbf{B}}{\partial t} d(\Delta V) = \frac{\partial \mathbf{B}_c}{\partial t} \Delta V \quad (23)$$

where

$$\Delta V = \frac{1}{3} \int_{\Delta S} (\mathbf{x} \cdot \mathbf{n}) d(\Delta S) \quad (24)$$

In the newly proposed method, the integration is performed directly by using Relation (21) as follows:

$$\int_{\Delta V} \frac{\partial B_i}{\partial t} d(\Delta V) = \int_{\Delta V} \frac{\partial}{\partial x_j} \left( \frac{\partial B_j}{\partial t} x_i \right) d(\Delta V) = \int_{\Delta S} \left( \frac{\partial B_j}{\partial t} n_j \right) x_i d(\Delta S) \quad (25)$$

The same in the vector form:

$$\int_{\Delta V} \frac{\partial \mathbf{B}}{\partial t} d(\Delta V) = \int_{\Delta S} \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) \mathbf{x} d(\Delta S) \quad (26)$$

After the integration over the volume  $V$ , (22) can be easily written in the vector variant:

$$\int_S \left[ (\mathbf{n} \times \mathbf{E}) + \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) \mathbf{x} \right] dS = 0 \quad (27)$$

Note that from the obtained results (20), it is observable that the non-stationary change in magnetic induction within the field  $V$  can be determined by integration only at the system boundary. Conversely, time changes in the magnetic induction in the field  $V$  are generated at the system boundary. This fact can be advantageously used for both qualitative analysis of non-stationary boundary conditions and optimization of non-stationary tasks by selecting a suitable target function depending on the boundary conditions. For the non-conductive space, the derivations are described in Section 3.

### 3. A Non-Conductive Environment

We assume  $\sigma = 0$ ,  $\rho_e = 0$ . In this case, Maxwell's equations will have the index symbolic form [1,3]:

$$\varepsilon_{ijk} \frac{\partial H_k}{\partial x_j} = \frac{\partial D_i}{\partial t}, \quad \frac{\partial D_i}{\partial x_i} = 0 \quad (28)$$

$$\varepsilon_{ijk} \frac{\partial E_k}{\partial x_j} = -\frac{\partial B_i}{\partial t}, \quad \frac{\partial B_i}{\partial x_i} = 0. \quad (29)$$

Due to the validity of Equations (17) and (18) using transform (21), new forms of Maxwell's equations can be written without evidence, since:

$$\begin{aligned} \frac{\partial D_i}{\partial t} &= \frac{\partial}{\partial x_j} \left( \frac{\partial D_j}{\partial t} x_i \right) \\ \frac{\partial B_i}{\partial t} &= \frac{\partial}{\partial x_j} \left( \frac{\partial B_j}{\partial t} x_i \right) \end{aligned} \quad (30)$$

$$\begin{aligned} \int_V \frac{\partial D_i}{\partial t} dV &= \int_S \frac{\partial D_j}{\partial t} x_i n_j dS \\ \int_V \frac{\partial D}{\partial t} dV &= \int_S \left( \frac{\partial D}{\partial t} \cdot \mathbf{n} \right) \mathbf{x} dS \end{aligned} \quad (31)$$

$$\begin{aligned} \int_V \frac{\partial B_i}{\partial t} dV &= \int_S \frac{\partial B_j}{\partial t} x_i n_j dS \\ \int_V \frac{\partial B}{\partial t} dV &= \int_S \left( \frac{\partial B}{\partial t} \cdot \mathbf{n} \right) \mathbf{x} dS \end{aligned} \quad (32)$$

$$\begin{aligned} \frac{\partial}{\partial x_j} \left( \frac{\partial D_j}{\partial t} x_i - \varepsilon_{ijk} H_k \right) &= 0 \\ \frac{\partial}{\partial x_j} \left( \frac{\partial B_j}{\partial t} x_i + \varepsilon_{ijk} E_k \right) &= 0 \end{aligned} \quad (33)$$

$$\begin{aligned} \int_S \left[ \left( \frac{\partial D}{\partial t} \cdot \mathbf{n} \right) \mathbf{x} - \mathbf{n} \times \mathbf{H} \right] dS &= 0 \\ \int_S \left[ \left( \frac{\partial B}{\partial t} \cdot \mathbf{n} \right) \mathbf{x} + \mathbf{n} \times \mathbf{E} \right] dS &= 0 \end{aligned} \quad (34)$$

From the above, it is visible that in the case of a non-conductive environment, it is possible to derive a new variant for all Maxwell's equations. All the conclusions given in Section 3 remain valid, including the control volumes method.

All these results can be used to solve the interdisciplinary problem of the motion of an incompressible fluid with the effects of a nonconductive magnetic field using the Maxwell stress tensor for this case; see Section 6.

### 4. The Scalar Variants of Maxwell's Equations

By the scalar variant of Maxwell's equations [1,3], we mean the product of the multiplication of Equation (13) and the position vector  $\mathbf{x}$ .

$$\begin{aligned} \text{curl} \mathbf{E} \cdot \mathbf{x} &= -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{x}, \\ \varepsilon_{ijk} \frac{\partial E_k}{\partial x_j} x_i &= -\frac{\partial B_i}{\partial t} x_i \end{aligned} \quad (35)$$

The results of the solution of the scalar variant can be used again for the boundary conditions analysis and in the optimization area. For the solution, the divergence theorem (13) is beneficially used:

$$\text{div} \mathbf{B} = 0 \Rightarrow \frac{\partial B_j}{\partial x_j} x_i x_i = 0 \quad (36)$$

By analogy to (15), we apply the multiplication:

$$\begin{aligned} \frac{\partial B_j}{\partial x_j} x_i x_i &= 0; \\ \int_V \frac{\partial B_j}{\partial x_j} x_i x_i dV &= \int_S B_j x_i x_i n_j dS - 2 \int_V B_j \frac{\partial x_i}{\partial x_j} x_i dV \end{aligned} \quad (37)$$

From (37) follows the important knowledge:

$$\int_V B_j \delta_{ij} x_i dV = \int_V B_i x_i dV = \frac{1}{2} \int_S B_j x_i x_i n_j dS \quad (38)$$

In the vector form written as:

$$\int_V \mathbf{B} \cdot \mathbf{x} dV = \frac{1}{2} \int_S (\mathbf{B} \cdot \mathbf{n})(\mathbf{x} \cdot \mathbf{x}) dS \quad (39)$$

Considering the divergence theorem in the shape  $\text{div} \frac{\partial \mathbf{B}}{\partial t} = 0$ , Equations (38) and (39) can be written:

$$\int_V \frac{\partial B_i}{\partial t} x_i dV = \frac{1}{2} \int_S \frac{\partial B_j}{\partial t} x_i x_i n_j dS \quad (40)$$

$$\int_V \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{x} dV = \frac{1}{2} \int_S \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) (\mathbf{x} \cdot \mathbf{x}) dS \quad (41)$$

$$\text{where } (\mathbf{x} \cdot \mathbf{x}) = x_i x_i = |\mathbf{x}|^2$$

From Equation (40) and using the divergence theorem, it is possible to derive the following important dependence that allows one to reformulate Maxwell's Equation (35):

$$\frac{\partial B_i}{\partial t} x_i = \frac{1}{2} \frac{\partial}{\partial x_j} \left( \frac{\partial B_j}{\partial t} x_i x_i \right) \quad (42)$$

When we implement (42) into (35), we obtain:

$$\frac{\partial}{\partial x_j} \left( \varepsilon_{ijk} E_k x_i + \frac{1}{2} \frac{\partial B_j}{\partial t} x_i x_i \right) = 0 \quad (43)$$

The term in the bracelet can again be divided into the symmetric and antisymmetric parts as well as in (22). In the integral form:

$$\int_S \left[ \frac{1}{2} \frac{\partial B_j}{\partial t} x_i x_i + \varepsilon_{ijk} E_k x_i \right] n_j dS = 0. \quad (44)$$

For the vector form, it holds:

$$\int_S \left[ \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) (\mathbf{x} \cdot \mathbf{x}) + 2(\mathbf{n} \times \mathbf{E}) \cdot \mathbf{x} \right] dS = 0. \quad (45)$$

One of the scalar variants of Maxwell's equations was again derived under the assumption of Gauss's divergence theorem validity. Derived relationships can be used to evaluate the results obtained by numerical methods. Even in this case, non-stationary changes in magnetic induction are reflected at the boundary of the region, and, here, it is possible to determine their values as a function of time. The resulting equations can also be easily used for optimization because the target function is scalar in this case.

## 5. A Non-Conductive Environment—Scalar Variant

If we return to the problem of a non-conductive environment, we can rewrite Equations (28) and (29) in a differential form:

Original equations:

$$\begin{aligned} \frac{\partial D_i}{\partial t} x_i - \varepsilon_{ijk} \frac{\partial H_k}{\partial x_j} x_i &= 0 \\ \frac{\partial B_i}{\partial t} x_i + \varepsilon_{ijk} \frac{\partial E_k}{\partial x_j} x_i &= 0 \end{aligned} \quad (46)$$

$$\frac{\partial B_i}{\partial x_i} = 0 \quad (47)$$

New variant:

$$\begin{aligned} \frac{\partial}{\partial x_j} \left( \frac{\partial D_j}{\partial t} x_i x_i - 2\varepsilon_{ijk} \frac{\partial H_k}{\partial x_j} x_i \right) &= 0 \\ \frac{\partial}{\partial x_j} \left( \frac{\partial B_j}{\partial t} x_i x_i + 2\varepsilon_{ijk} \frac{\partial E_k}{\partial x_j} x_i \right) &= 0. \end{aligned} \quad (48)$$

Integral form:

Original equations:

$$\begin{aligned} \int_V \frac{\partial D}{\partial t} \cdot x dV &= \int_S (\mathbf{n} \times \mathbf{H}) \cdot x dS; \\ \int_V \frac{\partial B}{\partial t} \cdot x dV &= - \int_S (\mathbf{n} \times \mathbf{E}) \cdot x dS. \end{aligned} \quad (49)$$

New variant:

$$\begin{aligned} \int_S \left( \frac{\partial D}{\partial t} \cdot \mathbf{n} \right) (\mathbf{x} \cdot \mathbf{x}) dS &= 2 \int_S (\mathbf{n} \times \mathbf{H}) \cdot x dS; \\ \int_S \left( \frac{\partial B}{\partial t} \cdot \mathbf{n} \right) (\mathbf{x} \cdot \mathbf{x}) dS &= -2 \int_S (\mathbf{n} \times \mathbf{E}) \cdot x dS. \end{aligned} \quad (50)$$

The new form of Maxwell's equations is useful, whether for analysis or numerical solution. Comparing the left sides of Equations (49) and (50), it is obvious that non-stationary variables  $\mathbf{D}(\mathbf{x}, t)$ ,  $\mathbf{B}(\mathbf{x}, t)$ , are generated only at the boundary of the system, and, therefore, it is possible to influence their process within the volume  $V$ . This can be essential in optimizing the non-stationary problems of electromagnetism.

Here, scalar variants of Maxwell's equations were also derived under the assumption of Gauss's divergence theorem validity. Derived relations can be used to evaluate the results obtained by numerical methods. Even in this case, non-stationary changes in magnetic induction and electrical induction are reflected at the boundary of the region, and here, their values can be determined as a function of time. The resulting equations can also be easily used for optimization because the target function is scalar in this case.

## 6. An Interaction of a Non-Conductive Magnetic Liquid with a Magnetic Field

In Section 3, it is shown that in the case of a non-conductive environment, it is possible to derive a new variant for all Maxwell's equations. All the conclusions given in the Section remain valid, including the control volumes method. All these results can be used in solving the interdisciplinary problem of the motion of an incompressible fluid with the effects of a non-conductive magnetic field using the Maxwell stress tensor for this case [3]. In the presented case, we assume:

$$\rho_e = 0, \quad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}, \quad \mathbf{M} = \chi \mathbf{H}. \quad (51)$$

The density of the volumetric magnetic force that acts on the elementary volume can be written in the form [8]:

$$\mathbf{f} = \frac{1}{2} \chi \text{grad} H^2; \quad H^2 = \mathbf{H} \cdot \mathbf{H}. \quad (52)$$

Navier–Stokes equations of the magnetic liquid in the presented case are in the form [2,11,25–27]:

$$\rho \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} (v_i v_j - \sigma_{ij}) = \rho g_i + \frac{1}{2} \chi \frac{\partial}{\partial x_i} (H^2). \quad (53)$$

Considering

$$g_i = \frac{\partial}{\partial x_i} (g_k x_k), \quad (54)$$



then, Equation (53) can be written in a more transparent form:

$$\rho \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho v_i v_j - \sigma_{ij} - \rho \delta_{ij} g_k x_k - \frac{1}{2} \chi \delta_{ij} H^2 \right) = 0. \quad (55)$$

Because the liquid is considered to be incompressible, the continuity equation is in the form:

$$\frac{\partial v_i}{\partial x_i} = \text{div} \mathbf{v} = 0. \quad (56)$$

Now, if we consider Equation (21), the Navier–Stokes equation for the incompressible magnetic liquid can be written in a new form:

$$\frac{\partial}{\partial x_j} \left[ \rho \frac{\partial v_j}{\partial t} x_i + \rho v_i v_j - \sigma_{ij} - \rho \delta_{ij} \left( g_k x_k - \frac{1}{2} \chi H^2 \right) \right] = 0. \quad (57)$$

This equation can be, using the Divergence theorem [16,17], rewritten in the new integral form:

$$\int_S \left[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{n} \right) x + \rho (\mathbf{v} \cdot \mathbf{n}) \mathbf{v} - \boldsymbol{\sigma} - \left( \mathbf{g} \cdot \mathbf{x} - \frac{1}{2} \chi (\mathbf{H} \cdot \mathbf{H}) \right) \cdot \mathbf{n} \right] dS = 0 \quad (58)$$

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3); \quad \sigma_i = \sigma_{ij} n_j. \quad (59)$$

By comparing the original equation, Equation (53), and the new equation, Equation (57), the advantage of the new variant is evident both for the numerical solution by the finite volume method and for the analysis of the influence of boundary conditions. Since in the above-mentioned case, assuming  $\text{div} \mathbf{v} = 0$ , with respect to (20), it holds:

$$\int_V \frac{\partial \mathbf{v}}{\partial t} dV = \int_S \left( \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{n} \right) x dS \quad (60)$$

$$\int_V \frac{\partial \mathbf{B}}{\partial t} dV = \int_S \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) x dS \quad (61)$$

and concurrently for the result of the continuity equation:

$$\int_S \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{n} dS = 0, \quad \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS = 0$$

## 7. Discussion

A new formulation of Maxwell's equations was derived, both in differential and integral variants. The basis for the derivation was Gauss's divergence theorem, used for magnetic flux density  $\mathbf{B}$  and electric flux density  $\mathbf{D}$ . By the use of Gauss's divergence theorem, Maxwell's equations were transformed. This resulted in a tool that can be used in the numerical finite volume method and optimization. The obtained equations will also allow the qualitative analysis of the influence of boundary conditions. The mentioned changes concern the non-stationary terms of the type  $\frac{\partial \mathbf{B}}{\partial t}$ , resp.  $\frac{\partial \mathbf{D}}{\partial t}$ . This resulted in a new form of Maxwell's equations that can be used in solving the interdisciplinary problem of the motion of an incompressible fluid with the effects of a non-conductive magnetic field using the Maxwell stress tensor for this case. For example:

### Differential forms

Original Maxwell's equation:

$$\frac{\partial B_i}{\partial t} = -\varepsilon_{ijk} \frac{\partial E_k}{\partial x_j},$$

New variant:

$$\frac{\partial}{\partial x_j} \left( \frac{\partial B_j}{\partial t} x_i \right) = -\varepsilon_{ijk} \frac{\partial E_k}{\partial x_j}.$$

### Integral forms

Original variant:

$$\int_V \frac{\partial \mathbf{B}}{\partial t} dV = - \int_S \mathbf{n} \times \mathbf{E} dS,$$

New variant:

$$\int_S \left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) x dS = - \int_S \mathbf{n} \times \mathbf{E} dS.$$

In these relations, it is interesting that the effect of non-stationary members  $\frac{\partial \mathbf{B}}{\partial t}$  within the region  $V$  is reflected at the system boundary only by its normal component  $\left( \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right)$ .

## 8. Conclusions

The work was focused on the analysis of non-stationary Maxwell equations. A new shape of non-stationary magnetic flux density was derived. This made the analyses of Maxwell's equations possible by using the Gaussian divergence theorem. Maxwell's equations were defined in both vector and scalar variants. The new shape of the Maxwell equations simplifies the analyses of the solution quality depending on the boundary conditions, considering the non-stationary magnetic induction. It also allows the numerical solution of Maxwell's equations to be extended to the large control volume method. Using the Gaussian divergence theorem, the new method allows the region to be optimized depending on the non-stationary field of magnetic induction.

A special part was devoted to the finite volume method for non-stationary problems. In the classical method, a non-stationary term is identified through the control volume in terms of the mean values of the integral calculus. This method does not allow the use of large control volumes, while the new variant allows it.

Both conductive and non-conductive environments were considered in the solution. The last Section presents a mathematical model of the interaction of a magnetorheological fluid with a magnetic field. Even for this interdisciplinary problem, Gauss's divergence theorem can be used to redefine the mathematical model of Navier–Stokes equations.

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**Notes:** Einstein summation convention is used in the article.

## Nomenclature

$x_i$	Cartesian coordinates
$t$	time
$V$	volume
$\Delta V$	control volume
$\Delta S$	control surface
$S$	closed surface
$\hat{S}$	open surface
$\mathbf{x} = (x_1, x_2, x_3)$	spatial vector
$\mathbf{n} = (n_1, n_2, n_3)$	unit normal vector
$\mathbf{E}$	electric field intensity
$\mathbf{D}$	electric flux density
$\mathbf{j}$	current density
$\rho_e$	charge density
$\mathbf{B}$	magnetic flux density
$\mathbf{H}$	magnetic field intensity
$\sigma$	conductivity
$\epsilon$	permittivity
$\mathbf{M}$	magnetization
$\rho$	fluid density
$\mathbf{v}$	fluid velocity $\mathbf{v} = (v_1, v_2, v_3)$
$\mathbf{g}$	gravity acceleration
$\mathbf{y} \cdot \mathbf{z} = y_i z_i$	scalar product of two vectors $\mathbf{y}, \mathbf{b}$
$\sigma_i$	stress vector
$\sigma_{ij}$	stress tensor
$\delta_{ij}$	Kronecker delta
$\epsilon_{ijk}$	Levi-Civita tensor
$\chi$	magnetic susceptibility
$\mu_0$	surroundings permeability

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